3.

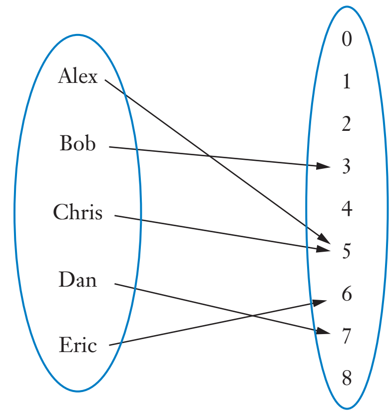
Function

- So exactly what do we mean by 'a function'?
- Machine analogy
- Natural domain of a function
- Miscellaneous exercise three

So exactly what do we mean by 'a function'?

WS
Functions and relations

Five soccer players took part in a penalty taking competition. Each player took eight penalties and the arrow diagram on the right shows the number of goals each player scored from these eight penalties.



If we select a name from the first set, the arrow diagram indicates the number of goals that player scored.

As was mentioned in the *Preliminary work* section at the beginning of this text, any relationship that assigns to each element of one set one *and only one* element from a second set is called a **function**.

We say that an element of the first set **maps onto** an element of the second set. For example, Alex in the first set maps onto 5 in the second set. We write Alex $\mapsto$ 5.

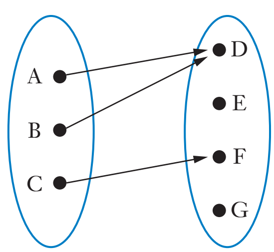
The first set we call the **domain**, the second set we call the **co-domain** and those elements of the co-domain that the elements of the first set map onto form the **range**.

Thus in the above function, the domain is {Alex, Bob, Chris, Dan, Eric},
the co-domain is {0, 1, 2, 3, 4, 5, 6, 7, 8}
and the range is {3, 5, 6, 7}.

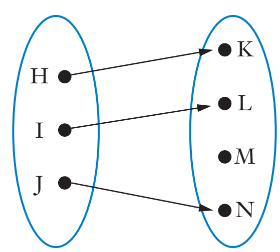
Notice that in this function two elements of the domain, Alex and Chris, map onto the same element of the range, 5. We call such functions, in which more than one element of the domain map onto the same element of the range, **many-to-one** functions.

If each element of the domain is mapped onto a different element of the range then the function is said to be **one-to-one**.

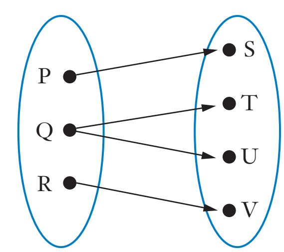
One-to-many relationships can occur but under our requirement that a function takes one element from the domain and assigns to it one *and only one* element from the range, a one-to-many relationship would not be called a function. (Thus the arrow diagram for a function cannot have any elements in the first set from which more than one arrow leaves.) This terminology is further illustrated below:



A many-to-one function
Domain {A, B, C}
Co-domain {D, E, F, G}
Range {D, F}



A one-to-one function
Domain {H, I, J}
Co-domain {K, L, M, N}
Range {K, L, N}



A 'one to many' relationship
Therefore not a function.

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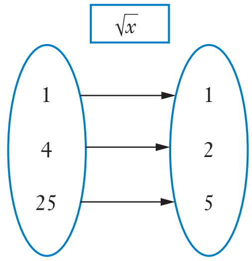
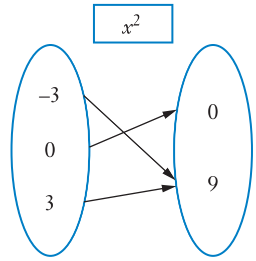
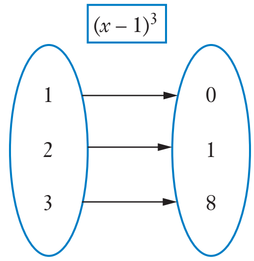
Many mathematical functions can be formed using a calculator. The correct key strokes can perform certain functions on a given number and give the appropriate output. For example:





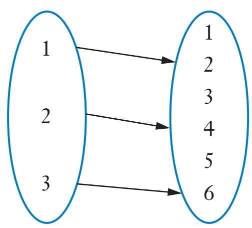


These functions may be *one-to-one*, as in $\sqrt{x}$ and $(x-1)^3$ shown above, or they may be *many-to-one*, as in x^2 shown above, for which $(-3)^2$ and $(3)^2$ have the same output, 9.

The domain and range of most of the functions we will deal with will be sets of numbers.

For example:

The '× 2' function
with domain {1, 2, 3}



Range {2, 4, 6}.

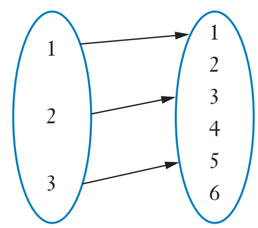
We write: $f(1) = 2$

$f(2) = 4$

$f(3) = 6$

Thus $f(x) = 2x$

The '× 2 and subtract 1' function
with domain {1, 2, 3}



Range {1, 3, 5}.

We write: $f(1) = 1$

$f(2) = 3$

$f(3) = 5$

Thus $f(x) = 2x - 1$

When several *functions* are used in one question, $g(x)$ and $h(x)$ are commonly used to distinguish between them.

EXAMPLE 1

If $f(x) = 5x + 1$ and $g(x) = x^2 - 5$ determine **a** $f(3)$ **b** $g(-2)$ **c** p given that $f(p) = g(p)$

Solution

a $f(x) = 5x + 1$

$\therefore\ f(3) = 5(3) + 1$

$= 16$

Thus $f(3)$ is 16.

b $g(x) = x^2 - 5$

$\therefore\ g(-2) = (-2)^2 - 5$

$= -1$

Thus $g(-2)$ is –1.

c $f(p) = g(p)$

$\therefore\ 5p + 1 = p^2 - 5$

$0 = p^2 - 5p - 6$

$0 = (p + 1)(p - 6)$

$p = -1$ or 6

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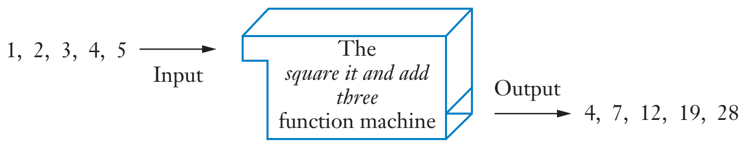
Machine analogy

As the *Preliminary work* reminded us it can be useful at times to consider a function as a machine. A box of numbers (the domain) is fed into the machine, a certain rule is applied to each number, and the resulting output forms a new box of numbers, the range.

In this way $f(x) = x^2 + 3$, with domain {1, 2, 3, 4, 5}, could be 'pictured' as follows:



Natural domain of a function

If we are not given a specific domain we assume it to be all the numbers which the function 'can cope with', i.e. all of the real numbers for which the function is *defined*. We call this the **natural domain** or **implied domain** of the function.

For example	$f(x) = 2x + 3$	is defined for all real x.
Thus	$f(x) = 2x + 3$	has natural domain $\mathbb{R}$.

(As mentioned in the preliminary work section we use $\mathbb{R}$, or **R**, to represent the set of all real numbers.)

	$g(x) = \dfrac{1}{x-3}$	is not defined for $x = 3$.
Thus	$g(x) = \dfrac{1}{x-3}$	has natural domain $\{x \in \mathbb{R}: x \neq 3\}$.

Reading '$\in$' as 'is a member of' and ':' as 'such that', $\{x \in \mathbb{R}: x \neq 3\}$ can be read as *x is a member of the set of real numbers such that x is not equal to 3*.

	$h(x) = \sqrt{x-3}$	is only defined for $x - 3 \geq 0$
Thus	$h(x) = \sqrt{x-3}$	has natural domain $\{x \in \mathbb{R}: x \geq 3\}$.

EXAMPLE 2

State the range of each of the following functions for the given domain.

a $f(x) = x + 1$ $\{x \in \mathbb{R}: 2 \leq x \leq 5\}$

b $f(x) = \sqrt{x}$ for the natural domain of $f(x)$.

Solution

a Adding 1 to all the real numbers from 2 to 5 will give all the real numbers from 3 to 6.
Thus the range is $\{y \in \mathbb{R}: 3 \leq y \leq 6\}$.

Note that we could use any letter to define the range but in this book we will tend to use x as the variable when defining a domain and y as the variable when defining a range.

b The natural domain of the function is $\{x \in \mathbb{R}: x \geq 0\}$.
This function could then output any non negative real number.
Thus the range is $\{y \in \mathbb{R}: y \geq 0\}$.

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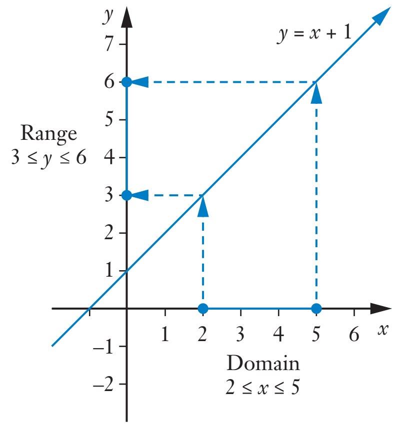
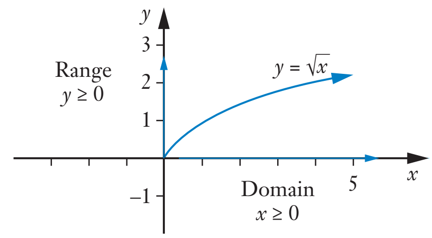
Alternatively Example 2 could be considered graphically. If we view the graph $y = f(x)$ then the required range will be the y values corresponding to the x values in the domain.

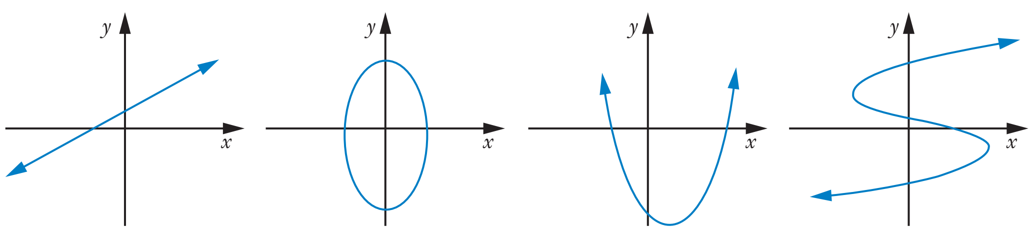




Note that with our requirement that a function takes one element from the domain and assigns to it one *and only one* element from the range, the graph of a function must be such that:

If a vertical line is moved from the left of the domain to the right it must never cut the graph in more than one place.

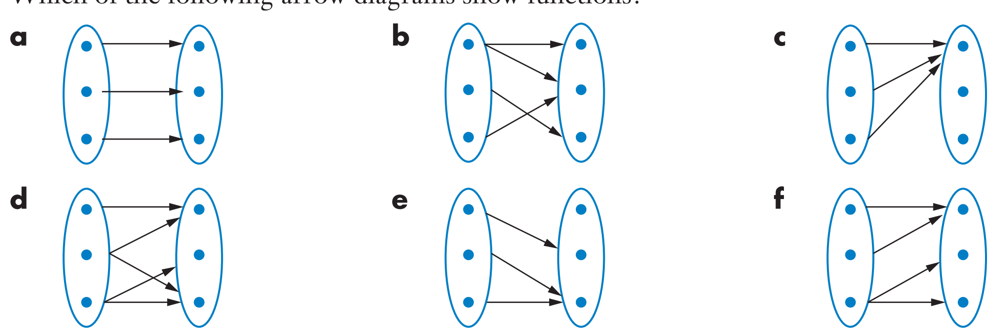
This is called the **vertical line test**.



Note: We could use a similar *horizontal line test* to determine whether a function is a one-to-one function or not.

Exercise 3A

1 Which of the following arrow diagrams show functions?



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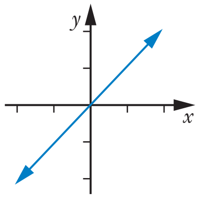
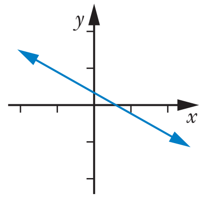
2 Which of the following shows the graph of a function?

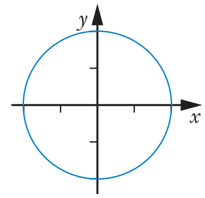
a



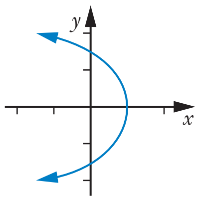
b



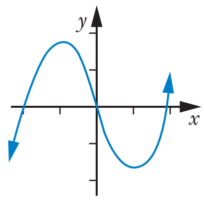
c



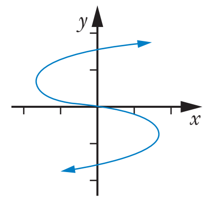
d



e

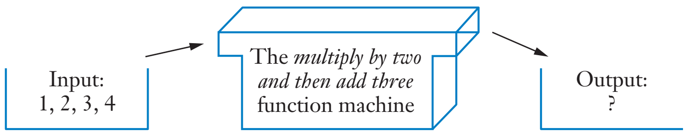


f

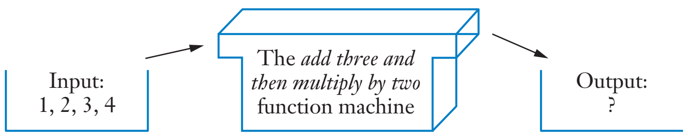


3 State the range of each of the following function machines for the domains shown.

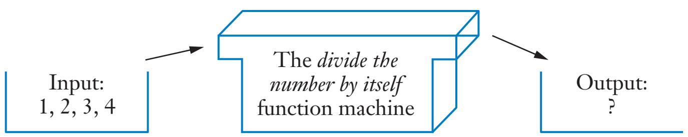
a



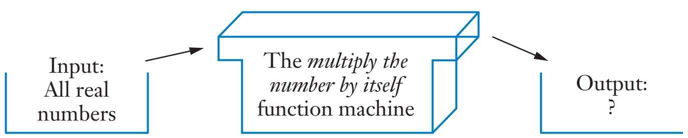
b



c



d



4 For $f(x) = 5x - 2$ determine each of the following.

a $f(4)$ **b** $f(-1)$ **c** $f(3)$ **d** $f(1.2)$

e $f(3) + f(2)$ **f** $f(5)$ **g** $f(-5)$ **h** $f(a)$

i $f(2a)$ **j** $f(a^2)$ **k** $3f(2)$ **l** $f(a + b)$

m The value of p for which $f(p) = 33$.

n The value of q for which $f(q) = -12$.

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5 For $f(x) = 4x - 7$, $g(x) = x^2 - 12$ and $h(x) = x^2 - 3x + 3$, determine each of the following.

a $f(4)$ **b** $f(0)$ **c** $g(3)$ **d** $g(-3)$

e $h(-5)$ **f** $h(5)$ **g** $h(-2)$ **h** $3f(a)$

i $f(3a)$ **j** $3g(a)$ **k** $g(3a)$

l The values of p for which $g(p) = 24$.

m The value of q for which $g(q) = h(q)$.

n The values of r for which $h(r) = f(r) + 28$.

6 Which numbers can each of the following functions **not** cope with? (i.e. which numbers must not be included in the domain?)

a $f(x) = \sqrt{x-1}$ **b** $f(x) = x^2 + 1$ **c** $f(x) = \dfrac{1}{x}$ **d** $f(x) = \dfrac{1}{1-x}$

7 Which numbers is it impossible for each of the following functions to output? (i.e. which numbers will not be included in the range?)

a $f(x) = \sqrt{x-1}$ **b** $f(x) = x^2 + 1$ **c** $f(x) = \dfrac{1}{x}$ **d** $f(x) = \dfrac{1}{1-x}$

For questions **8** to **22** state the range of each function for the given domain.

8 Function: $f(x) = x + 5$, Domain: $\{x \in \mathbb{R}: 0 \le x \le 3\}$

9 Function: $f(x) = x - 3$, Domain: $\{x \in \mathbb{R}: 0 \le x \le 3\}$

10 Function: $f(x) = 3x$, Domain: $\{x \in \mathbb{R}: -2 \le x \le 5\}$

11 Function: $f(x) = 4x$, Domain: $\{x \in \mathbb{R}: 5 \le x \le 10\}$

12 Function: $f(x) = 2x - 1$, Domain: $\{x \in \mathbb{R}: 0 \le x \le 5\}$

13 Function: $f(x) = 1 - x$, Domain: $\{x \in \mathbb{R}: 0 \le x \le 5\}$

14 Function: $f(x) = x^2$, Domain: $\{x \in \mathbb{R}: -1 \le x \le 3\}$

15 Function: $f(x) = (x + 1)^2$, Domain: $\{x \in \mathbb{R}: -2 \le x \le 3\}$

16 Function: $f(x) = x^2 + 1$, Domain: $\{x \in \mathbb{R}: -1 \le x \le 3\}$

17 Function: $f(x) = \dfrac{1}{x}$, Domain: $\{x \in \mathbb{R}: 1 \le x \le 4\}$

18 Function: $f(x) = \dfrac{1}{x}$, Domain: $\{x \in \mathbb{R}: 0 < x \le 1\}$

19 Function: $f(x) = x^2 - 1$, Domain: $\mathbb{R}$

20 Function: $f(x) = x^2 + 4$, Domain: $\mathbb{R}$

21 Function: $f(x) = \dfrac{1}{x-1}$, Domain: $\{x \in \mathbb{R}: x \ne 1\}$

22 Function: $f(x) = \dfrac{x+1}{x-1}$, Domain: $\{x \in \mathbb{R}: x \ne 1\}$

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For questions **23** to **28** state whether the function is one-to-one or many-to-one for the stated domain.

23 $f(x) = x$, domain: $\mathbb{R}$

24 $f(x) = x^2$, domain: $\{x \in \mathbb{R}: 0 \le x \le 3\}$

25 $f(x) = x^2$, domain: $\{x \in \mathbb{R}: -3 \le x \le 3\}$

26 $f(x) = x^2$, domain: $\mathbb{R}$

27 $f(x) = \sqrt{x}$, domain: $\{x \in \mathbb{R}: 1 \le x \le 4\}$

28 $f(x) = \sqrt{x}$, for the natural domain of the function

State the natural domain and corresponding range for each of the following.

29 $f(x) = 2x + 3$

30 $f(x) = x^2$

31 $f(x) = \sqrt{x}$

32 $f(x) = \sqrt{x-3}$

33 $f(x) = \sqrt{x+3}$

34 $f(x) = 5 + \sqrt{x-3}$

35 $f(x) = \dfrac{1}{x-3}$

36 $f(x) = \dfrac{1}{\sqrt{x-3}}$

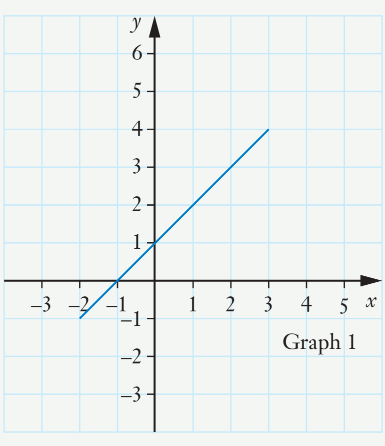
Miscellaneous exercise three

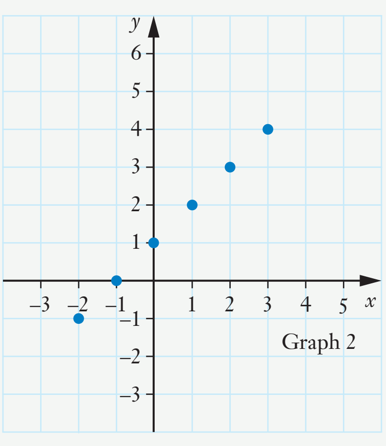
This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the Preliminary work section at the beginning of the book.

1 Solve **a** $\dfrac{2x-1}{3} = \dfrac{3x+2}{5}$ **b** $\dfrac{3x-1}{2} + 7 = \dfrac{2x+7}{3}$

2 Find the range of the function $f(x) = 3 - 2x$ for the domain $\{1, 2, 3, 4\}$.

3 Graph 1 below shows the graph of $f(x) = x + 1$ for the domain $-2 \le x \le 3$.
Graph 2 shows the graph of $f(x) = x + 1$ for the domain $\{-2, -1, 0, 1, 2, 3\}$.
State the range of the function for each domain.



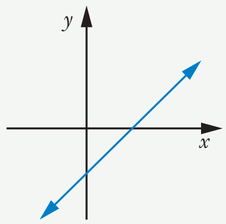


4 Expand and then simplify each of the following:

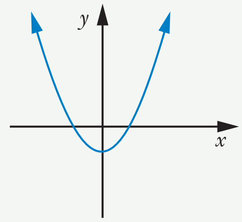
a $(a+b)^2$ **b** $(a+b)^3$ **c** $(a+2b)^3$ **d** $(a-2b)^3$

5 For each of the following diagrams, state whether the relationship shown is a function or not and, for those that are functions, state whether the function is one-to-one or many-to one.

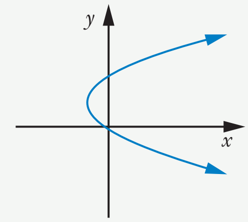
a



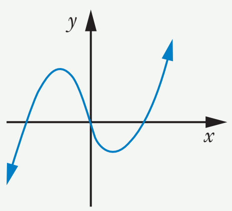
b



c



d



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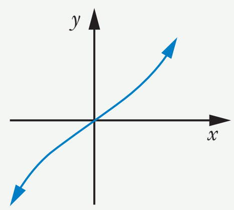

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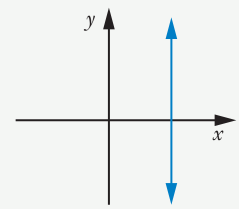
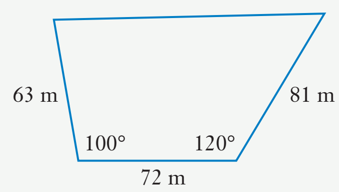


6 The isosceles triangle ABC has AB = AC = 10 cm and BC = 12 cm. Three circles are drawn, one with centre A, radius 4 cm, another with centre B, radius 6 cm, and a third with centre C, radius 6 cm. Find the area of that part of triangle ABC not lying in any of the circles.

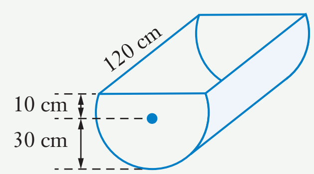
7 Three ships A, B and C are such that B is 4.8 km from A on a bearing 115° and C is 5.7 km from A on a bearing 203°.

How far and on what bearing is B from C?

8 The diagram on the right shows the sketch made by a surveyor after taking measurements for a block of land. Find the area of the block.



9 A pig farmer makes a feeding trough for his pigs by cutting a cylindrical metal drum (see diagram). If the drum is of length 120 cm and radius 30 cm and the cut is made 10 cm from the axis of the drum find the capacity of the trough correct to the nearest litre. (Ignore the thickness of the metal.)



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